12 Nonlinear Dynamics/Dynamical Systems

12.3 The Lorenz Equations (10 units)

Some familiarity with the Part II course Dynamical Systems would be helpful for this project, which is concerned with bifurcations and chaos in ordinary differential equations.

1 The Lorenz equations

The Lorenz equations are named after the meteorologist who first studied them in 1963:

$$\begin{aligned} \dot{x}(t) &= f_1(x,y,z) &&= 10(y-x)\ , \\ \dot{y}(t) &= f_2(x,y,z) &&= rx - y - xz\ , \\ \dot{z}(t) &= f_3(x,y,z) &&= xy - 8z/3\ . \end{aligned}$$

Question 1 Integrate the equations for values of $r = 0$, 15, 21 and 29. Use $x = y = 1$, $z = r - 1$ as the initial conditions. You may use any standard integrating packages that are available and enable you to choose an appropriate step-length and then fix on it*; comment however on the effect of changing the step-length and why you chose your particular value. You should plot $x(t)$ against $z(t)$ to show your results. For some of the above values of r consider plotting the solution only for $t > T$ for some time $T > 0$; why can this be useful?

A stationary point is a point (x, y, z) where $\dot{x} = \dot{y} = \dot{z} = 0$. It is (locally) stable if all the eigenvalues of the Jacobian matrix $Df(x) = (\partial f_i/\partial x_j)_{1 \leqslant i,j \leqslant 3}$ evaluated at the stationary point have negative real part.)

Question 2 Investigate analytically the existence and stability of stationary points of the flow. How do these results relate to the behaviour observed in question 1 above?

2 The strange attractor

The persistent erratic non-periodic oscillations seen when $r = 27$ are due to the existence of a "strange attractor" in the flow. (The existence of this attractor was discovered numerically by Lorenz but there is still no completely rigorous proof that it exists and has the properties we are about to study). This attractor is stable for (approximately) $r > 24.06$, but for $r < 24.06$ some solutions spend a long time wandering about near it before eventually tending towards a stable stationary point. (It exists, but is unstable, for approximately $13.9236 < r < 24.06$.) This phenomenon is known as *intermittency.*

Question 3 For various initial conditions as given in the following list, plot $x(t)$ against t at r-values of your choice in $23 < r < 25$: in each case include in your write-up one or two plots showing the different possible behaviours.

(i) Start very close to the origin $(0, 0, 0)$ but not on the z-axis (why not?).

(ii) Start very close to one of the other fixed points.

*For example you can download a suitable solver from http://www.mathworks.com/matlabcentral/answers/98293

(iii) Start near $x = y = 1$, $z = r - 1$.

Which type of initial condition is best suited for deciding the r-value at which the strange attractor becomes attracting? Which is useful to confirm your stability analysis for the stationary points obtained in question 2 above? Which best illustrates intermittency?

Question 4 For initial conditions which produce trajectories displaying intermittent behaviour in $r < 24.06$, plot the time spent wandering erratically before the trajectory spirals steadily into one of the stationary points against $(24.06 - r)$. You should decide on some criterion for deciding the time t_c at which the solution you are calculating starts heading towards a stable stationary point, and calculate the average of the values of t_c obtained for 5 different initial conditions (all of which should display several "erratic" oscillations before t_c is reached) at each r-value. Explain how you determine t_c.

You will find that nearby initial conditions sometimes give very different values of t_c, and as r increases towards 24.06 you may find that it becomes increasingly difficult to find t_c for all of your chosen initial conditions; you should start with $r = 20$ and be prepared to stop increasing r when the amount of machine time used becomes excessive.

Question 5 Suggest a formula for the way in which the average t_c value increases with r. You will need a fairly large sample of t_c values to make a reasonable estimate.

Question 6 For $r = 27$, write a program to record the successive z-values z_1, z_2, z_3, ... at which a trajectory achieves a local maximum in z. Plot these on a scatter diagram of z_{n+1} against z_n; include also for reference the diagonal line $z_{n+1} = z_n$. What property do portions of the trajectory which generate high points (large values of z_{n+1}) on this diagram have? Does the information that the origin $(0, 0, 0)$ is actually part of the strange attractor help you to find a numerical method to compute (approximately) the largest value of z_{n+1} that could appear on this diagram? If so, do it and add an appropriate point to your figure.

You should not plot the first few points obtained from any given trajectory in order to give any transient behaviour time to die out. You may generate points from many trajectories or from one long trajectory. You will observe that the points on this scatter diagram all lie very near to a certain curve C, which can therefore be used as a predictor for the successive z_i values.

Question 7 Describe in some detail the chief features of this curve and how they relate to your numerical solutions. In particular, consider the following points:

- Where are the intersections of C with the diagonal?
- Does C intersect the diagonal at $z = r - 1$?
- On your diagram, is it possible to draw a square whose top-right and bottom-left corners lie on the diagonal, whose top side touches the peak of C, whose bottom-right corner lies on C, but whose bottom edge does not otherwise intersect C?

Question 8 On a copy of your diagram, draw an approximation to the curve C and use this hand-drawn curve (which you should include in your write-up) to predict a succession of z_i values. For how many steps does your prediction agree well with an actual sequence produced by the numerically computed trajectory? Are there any features of the curve which would lead you to expect this result?

3 The effect of varying r

Question 9 How does the curve C vary as r decreases? Draw the curves for $r = 24.3$ and 22.9, extending them in a sensible way to $z = r - 1$. (For $r = 22.9$ you will need to use initial conditions which give intermittent trajectories in order to generate much of the curve). Describe how the features of the curve change, and explain how these changes relate to the other aspects of behaviour studied in this project.

References

[1] Colin Sparrow *The Lorenz equations: bifurcations, chaos, and strange attractors*. Springer, 1982.

