12 Nonlinear Dynamics/Dynamical Systems

12.9 Differential Equations for Nonlinear Oscillators (10 units)

Material in the Part II course Dynamical Systems is relevant to this project.

Introduction

Many nonlinear differential equations arise in physical, biological and chemical contexts. Several of these describe nonlinear oscillators and exhibit interesting dynamics.

The general equation of a forced nonlinear oscillator can be written as

$$\ddot{x} + (\alpha + \beta x^m)\dot{x} - \gamma x + \delta x^n = f(t)$$

When $\beta = 0$ and $n = 3$ the system described by this equation is known as a Duffing oscillator. When $\delta = 0$ and $m = 2$ it is know as a van Der Pol oscillator.

Part 1

We shall study the forced Duffing equation with periodic forcing in the form

$$\ddot{x} + a\dot{x} - x + x^3 = b\cos t$$

where a and b are constants and dot signifies differentiation with respect to t.

> **Question 1** Write a program (using the Runge-Kutta routine for example) to integrate this system from five initial conditions with $-2 \leqslant x(0) \leqslant 2$ and $-2 \leqslant \dot{x}(0) \leqslant 2$, plotting all five solutions on a single picture. (Plot $x(t)$ against $\dot{x}(t)$.)

> **Question 2** Test your program by running it with $b = 0$ at $a = -0.12$, $a = 0$ and $a = 0.12$ and show and describe the results. Comment on any special features of the case $a = b = 0$.

Now set $a = 0.15$ and $b = 0.3$. Use your program to find two stable solutions (one is a periodic orbit, the other looks like a "strange attractor", so solutions on this attractor never appear to settle down to any simple closed loop) which both exist at these parameter values.

> **Question 3** Choose two initial conditions, one which tends towards each attractor, and adapt your program to integrate with these two sets of initial conditions only. Display your results.

> **Question 4** Repeat this numerical experiment with the same two initial conditions and the same parameter values, but this time, instead of drawing the whole solution, plot points (without joining them up) only when $t = 2n\pi \quad (n = 0, 1, 2, \ldots)$. Comment on the relationship between the two different ways (whole trajectories and points) of representing solutions.

> **Question 5** Use this program to investigate in detail the behaviour of the system at different values of a between 0.1 and 0.5 (with $b = 0.3$ and a range of initial conditions), in particular the evolution of the strange attractor (when it exists). Show any pictures that seem interesting (four extra pictures are sufficient).

Part 2

Consider the equation

$$\ddot{x} + (x^2 - b)\dot{x} - ax + x^3 = 0.$$

For a and b small the parameter space can be divided into six regions as shown in Figure 1 overleaf. A choice of (a, b) in each of these six regions yields qualitatively different behaviour of solutions. Figure 1 remains a good approximate description of the regions and boundaries for moderate values of a, b, e.g. $|a| \leqslant 2$, $|b| \leqslant 2$.

Adapt the program of Part 1 to integrate these equations at given values of a and b with solutions from five different initial conditions displayed in a single picture.

> **Question 6** Run this program (with suitable choices of the initial conditions so that the pictures are as clear as possible) for a single value of (a, b) in each of the six regions, and show representative pictures for each region. You may like to set $b = \pm 1$ and vary a to find the different regions.

> **Question 7** Describe the dynamics in each region, including the nature of any fixed points or other features, and the transition from one region to the next. What happens as you move through the boundaries between each region? There is no need to find c; you are only required to find an example of behaviour in each region.

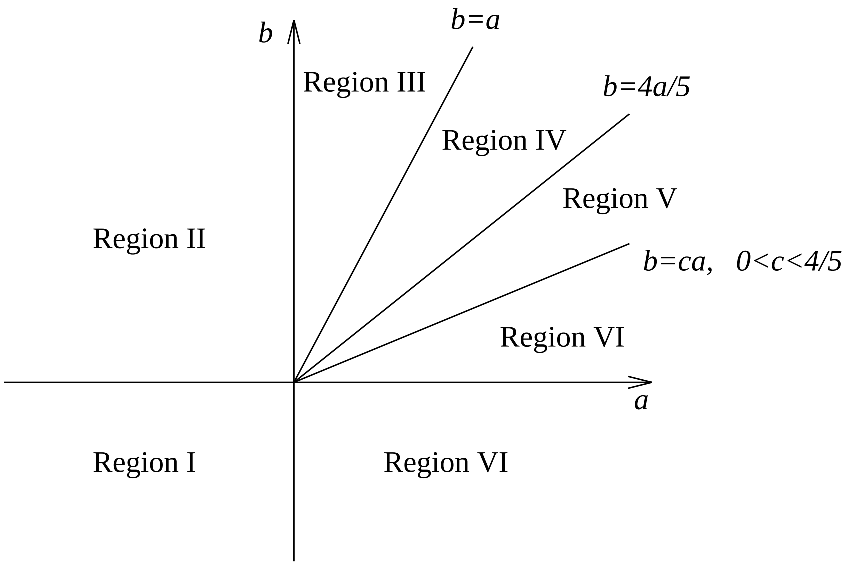

Figure 1

Part 3

The forced van der Pol oscillator

$$\ddot{x} + a(x^2 - 1)\dot{x} + x = 1 + b$$

has a Hopf bifurcation when $b = 0$. (In a Hopf bifurcation, a periodic orbit appears near a stationary point as the parameter, b, increases or decreases.) The above equation may be written in Liénard coordinates as $\dot{x} = y - a(x^3/3 - x)$, $\dot{y} = -x + 1 + b$. Adapt your program to integrate the forced van der Pol equation in this form.

Question 8 Locate the periodic orbit when $b = -0.001$ for $a = 1$, $a = 5$ and $a = 10$, giving a picture of the orbit by plotting $x(t)$ against $\dot{x}(t)$ at each time step.

Question 9 Investigate the evolution of the periodic orbit for $b \in [-0.1, 0)$ at each of these values of a, commenting on any unusual behaviour that you observe at particular values of b.

Question 10 Consider also the appearance of the orbit in the x-y plane. Can you explain its shape for large a? Explain what numerical difficulties can arise in calculating such an orbit.

References

[1] Drazin, P.G. *Nonlinear Systems.* CUP

[2] Glendinning, P. A. *Stability, Instability and Chaos.* CUP

