1 Numerical Methods

1.8 Hyperbolic Partial Differential Equations (8 units)

This project concerns shock formation and propagation in nonlinear hyperbolic equations. While this project is largely self-contained, knowledge of the Part II Waves and Part II Numerical Analysis courses is helpful. If you have not attended these courses then a relatively small amount of extra reading (e.g. as indicated in Question 1) will be necessary.

1 Background

The Euler equations of compressible fluid dynamics allow for the development of some interesting non-linear features; for example *shocks* (or sonic booms) and *rarefaction fans.* A useful model equation both for furthering our understanding of such solutions and for developing numerical methods for the Euler equations is

$$u_t + f(u)_x = 0, \qquad f(u) = \tfrac{1}{2}u^2\,, \tag{1}$$

known variously as the kinematic wave equation or the inviscid Burger's equation. This equation's interest lies in the fact it posesses a non-linear flux term proportional to the square of the basic variable u, identical to the convection term present in the Euler equations.

Question 1 Show analytically that u is constant along the characteristic curves of (1). Further, in the context of (1) and with the aid of characteristic diagrams (sketch only*), explain:

(i) the development of shocks, presenting
   (a) a derivation of the time at which a shock first occurs,
   (b) a statement and brief justification of the *Rankine-Hugoniot* relation for such a feature,
   (c) and thence a derivation of the speed at which the shock propagates;

(ii) the development of a rarefaction fan, mentioning the so-called *expansion shock* and why it is rejected as a physical solution.

You may find it helpful to refer to one or more of the textbooks by Billingham & King [1], Lighthill [2], Renardy & Rogers [3] or Whitham [5].

In the numerical work to follow we wish to solve (1) subject to the *discontinuous* initial condition

$$u(x,0) = \begin{cases} -\frac{1}{2} & x < \frac{1}{2}, \\ 1 & \frac{1}{2} \leqslant x < 1, \\ 0 & x \geqslant 1, \end{cases} \tag{2}$$

on the domain $x \in [0, 1.5]$. The boundary conditions at $x = 0$ and $x = 1.5$ should be taken to be those of *out-flow.*

* Whereas almost all graphs, including labels, annotations, etc., need to be computer-generated, this is one of the relatively few cases where a scanned hand-drawing **is** acceptable for electronic submission.

2 Numerical Fundamentals

We divide our domain of interest into J cells each of size Δx. Our basic variable u is conserved under the action of equation (1). It is, therefore, important that this is reflected in any numerical method we employ. A particular class of methods with this conservation property update the solution at the ith cell via

$$u_i^{n+1} = u_i^n + \frac{\Delta t}{\Delta x}\big(F(u_{i-1}^n, u_i^n) - F(u_i^n, u_{i+1}^n)\big). \tag{3}$$

In this equation n is the current time level, $n+1$ the next time level and Δt the time step between the two; $F(u_L, u_R)$ is the *numerical flux* through the *interface* between two neighbouring cells, u_L and u_R the states of the left and right cells respectively. The update is conservative no matter how we define the function F.

The time step Δt need not necessarily be constant throughout our numerical calculation. One approach is to let it be given by the formula

$$\Delta t = \frac{\Delta x \, C_{cfl}}{S_{\max}^n}, \tag{4}$$

wherein $0 < C_{cfl} \leqslant 1$ is the *Courant-Friedrichs-Lewy (CFL) coefficient* and $S_{\max}^n$ the maximum wave speed at time level n. In the case of equation (1) this is simply

$$S_{\max}^n = \max_{0 \leqslant i \leqslant J} \{|u_i^n|\}.$$

3 Numerical Methods

A simple numerical method is the Lax-Friedrichs scheme wherein the numerical flux is defined as

$$F^{\mathrm{LF}}(u_L, u_R) = \frac{1}{2}\big(f(u_L) + f(u_R)\big) + \frac{1}{2}\frac{\Delta x}{\Delta t}(u_L - u_R). \tag{5}$$

Question 2 Given the initial condition (2) and the boundary conditions specified in Section 1, write a program that marches equation (1) forward to a time $t = \frac{1}{2}$ using the Lax-Friedrichs scheme. Derive the exact solution analytically and compare your numerical solution with it. What is the order of accuracy of the scheme?

Another numerical method is the Richtmyer scheme. The numerical flux in this case is calculated via

$$u^{\mathrm{Ri}} = \frac{1}{2}(u_L + u_R) + \frac{1}{2}\frac{\Delta t}{\Delta x}\big(f(u_L) - f(u_R)\big), \qquad F^{\mathrm{Ri}}(u_L, u_R) = f(u^{\mathrm{Ri}}). \tag{6}$$

Question 3 Using the same initial and boundary conditions as in Question 2, write a program that marches equation (1) forward to a time $t = \frac{1}{2}$ using the Richtmyer scheme. Compare your numerical solution with the exact solution commenting on the order of accuracy and any interesting features you observe. How do these results compare with those obtained previously? Your remarks should include an outline discussion of the property known as *monotonicity* and a statement of its relevance to both the Richtmyer and Lax-Friedrichs schemes. Reference to the textbook by Toro [4] may be helpful.

In order to eradicate the inaccuracies associated with each of the above schemes, the following numerical flux was proposed:

$$F = F^{\mathrm{LO}} + \phi[F^{\mathrm{HI}} - F^{\mathrm{LO}}], \tag{7}$$

which contains both a high (HI) and low (LO) order flux and a *flux limiter* ϕ. The limiter lies in the range $0 \leqslant \phi \leqslant 1$ and acts to vary the overall flux, F, locally between low and high order.

There are many different limiter functions. We will consider here

$$\phi = \begin{cases} 0 & r \leqslant 0, \\ r & 0 \leqslant r \leqslant 1, \\ 1 & r \geqslant 1, \end{cases}$$

wherein r is defined locally as

$$r_i^n = \min\left\{ \frac{u_i^n - u_{i-1}^n}{u_{i+1}^n - u_i^n}, \frac{u_{i+2}^n - u_{i+1}^n}{u_{i+1}^n - u_i^n} \right\} \tag{8}$$

and is a measure of the local change in gradient of the solution.

Question 4 Is the overall flux, F, predominantly high or low order accurate in the neighbourhood of:

(i) sharp changes in gradient?

(ii) slight changes in gradient?

Outline your reasoning given the above formulae and why, in light of earlier results, such variation of F is desirable.

A particular method is the Flux LImited Centred (FLIC) scheme. In this case the high and low order fluxes are defined as

$$F^{\mathrm{HI}} = F^{\mathrm{Ri}},$$
$$F^{\mathrm{LO}} = \tfrac{1}{2}[F^{\mathrm{LF}} + F^{\mathrm{Ri}}].$$

Question 5 Using the same initial and boundary conditions as in Questions 2 and 3, write a program that marches equation (1) forward to a time $t = \frac{1}{2}$ using the FLIC scheme. Compare your numerical results with the exact solution and with the results obtained earlier. To what extent has the scheme eliminated the undesirable features of the previous methods?

References

[1] Billingham, J. & King, A.C. *Wave Motion: Theory and application*. Cambridge University Press, 2000.

[2] Lighthill, M.J. *Waves in Fluids*. Cambridge University Press, 1978.

[3] Renardy, M. & Rogers, R.C. *An Introduction to Partial Differential Equations*. Springer-Verlag, 1993.

[4] Toro, E.F. *Riemann Solvers and Numerical Methods for Fluid Dynamics*. Springer-Verlag, 1997.

[5] Whitham, G.B. *Linear and Nonlinear Waves*. Wiley, 1999.

