2 Waves

2.2 Dispersion (7 units)

This project assumes only the elementary properties of dispersive waves, covered in the Part II course Waves

1 Introduction

This project illustrates the way in which a disturbance in a dispersive system can change its shape as it travels. In order to fix ideas we will consider waves that are modelled by linear partial differential equations that (i) are second order in time, and (ii) have one spatial dimension.

We suppose that the disturbance is described by a function $F(x,t)$, where x is distance and t is time, e.g. F might represent the displacement to the free surface of a fluid. A formal solution for $F(x,t)$ can be obtained from the governing partial differential equation by the standard technique of taking a Fourier transform in the spatial direction. The general solution can then be written in the form

$$F(x,t) = \int_{-\infty}^{\infty} \left(A_1(k)\, \mathrm{e}^{-i\omega_1 t} + A_2(k)\, \mathrm{e}^{-i\omega_2 t} \right) \mathrm{e}^{2\pi i k x}\, dk \ , \tag{1}$$

where the amplitudes, A_1 and A_2, are fixed by the initial conditions, and the frequencies, $\omega_1(k)$ and $\omega_2(k)$, are solutions to the dispersion relation. We will assume that the model system is time-reversible so that $\omega_2(k) = -\omega_1(k)$.

For simplicity we will consider the case where, at $t = 0$,

$$F(x,0) = \frac{1}{\sqrt{\pi}} \exp\left(-\frac{x^2}{\sigma^2}\right) \quad \text{and} \quad \frac{\partial F}{\partial t}(x,0) = 0 \ . \tag{2}$$

The solution (1) then becomes

$$F(x,t) = \int_{-\infty}^{\infty} \hat{F}_0(k) \cos(\omega t)\, \mathrm{e}^{2\pi i k x}\, dk \ , \tag{3}$$

where we have written ω for ω_1, and

$$\hat{F}_0(k) = \int_{-\infty}^{\infty} F(x,0)\, \mathrm{e}^{-2\pi i k x} dx \ . \tag{4}$$

Question 1 If F is real, deduce a property of $\hat{F}_0(k)$.

In order to plot the above solutions some method is needed for evaluating the Fourier integrals in (3) and (4).

2 The Discrete Fourier Transform

Consider the Fourier Transform $\hat{G}(k)$ of a function $G(x)$:

$$\hat{G}(k) = \int_{-\infty}^{\infty} G(x)\, \mathrm{e}^{-2\pi i k x}\, dx \ . \tag{5}$$

Suppose that $G(x)$ is only appreciably non-zero over a limited range of x, say $-X/2 \leqslant x \leqslant X/2$, and that

$$G_r = G(r\Delta x) \quad \text{for} \quad r = -N/2, \ldots, (N/2-1) \ , \quad \text{where} \quad \Delta x = X/N \ . \tag{6}$$

Then an approximation to (5), known as the Discrete Fourier Transform (DFT), is

$$\hat{\mathcal{G}}_s = \frac{X}{N} \sum_{r=-N/2}^{N/2-1} G_r \, \Omega_N^{-rs} \quad \text{for} \quad s = -N/2, \ldots, (N/2-1) \ , \quad \text{where} \quad \Omega_N = \mathrm{e}^{2\pi i/N} \ . \tag{7}$$

The $\hat{\mathcal{G}}_s$ are approximations to the Fourier Transform at $k = s/X$.

The *exact* inverse of (7) is

$$G_r = \frac{1}{X} \sum_{s=-N/2}^{N/2-1} \hat{\mathcal{G}}_s \, \Omega_N^{rs} \ . \tag{8}$$

This is an approximation to

$$G(x) = \int_{-\infty}^{\infty} \hat{G}(k) \, \mathrm{e}^{2\pi i k x} dk \ , \tag{9}$$

based on the assumption that $\hat{G}(k)$ is only appreciably non-zero for $-N/2X \leqslant k \leqslant N/2X$. Hence the DFT and its inverse are asymptotic approximations under the dual limits $X \to \infty$ and $N/X \to \infty$.

3 The Fast Fourier Transform

The Fast Fourier Transform (FFT) technique is a quick method of evaluating sums of the form

$$\lambda_s = \sum_{r=0}^{N-1} \mu_r \, \Omega_N^{\sigma r s} \ , \quad s = 0, \ldots, N-1 \ , \quad \sigma = \pm 1 \ , \tag{10}$$

where the μ_r are a known sequence, and N is a power of a prime, or combination of primes; we will assume that N is a power of 2, say $N = 2^M$. A brief outline of the FFT is given in the appendix for reference, but it is not necessary to understand the details of the algorithm in order to complete the project. Indeed, for the computational part of this project you are advised to use a library FFT procedure, such as the Matlab routine `fft`. However, you are advised to note that sums are typically from 1 to N, while the theory in §2 has sums that run from $-N/2$ to $(N/2-1)$. This means that it may be necessary to re-position the part of the series with r, $s = N/2, \ldots, (N-1)$ to $-N/2, \ldots, -1$ (or vice versa), using the periodicity implicit in the definitions (7) and (8). Similar considerations also apply to available routines in other languages, and you should also in general take special care regarding sign conventions and scaling.

4 No Dispersion

Question 2 Write a program to draw graphs of F against x for various t.

Test it for times up to $t = 10$ for $\sigma = 2$ and the "dispersionless" dispersion relation $\omega^2 = 4\pi^2 k^2$. Choose appropriate values for the parameters X and N so that your plots are correct to "graphical accuracy"; present evidence of this accuracy in your write-up. Comment on your results.

5 Gravity Waves

Surface gravity waves in deep water have the dispersion relation

$$\omega^2 = 2\pi g|k|. \tag{11}$$

Question 3 For (11), and the values $\sigma = 10\,\mathrm{m}$ and $g = 9.81\,\mathrm{m\,s^{-2}}$ (in MKS units), draw graphs of F against x for various times.

- For $t = 2$ investigate the effects of changing the values of N and X (try starting with $N = 128$ and $X = 400$). Report the results of this investigation in your write-up, especially with respect to the errors in the solution, using both numerical values and plots. As regards understanding the behaviour of the solution for large $|x|$, you *may* find it helpful to evaluate an approximation to

$$\int_{-\infty}^{\infty} F(x,t)\,dx \tag{12}$$

  for different values of N and X.
- Plot the solution at times $t = 1, 2, \ldots, 6$. Arrange for solutions at more than one time to appear on the same plot. Include in your write-up a set of results that you have verified to be accurate, and that illustrate the behaviour of the disturbance best. Give justification for your choices of N and X. Comment on the features of the evolution, especially as related to the concept of group velocity. Can the concept of group velocity help you choose a suitable value of X for a given time?

6 Capillary Waves

Consider also the capillary-wave dispersion relation for surface waves where surface tension dominates over gravitational effects. This is given by

$$\omega^2 = \frac{8\pi^3\gamma}{\rho}|k|^3 , \tag{13}$$

where γ is the surface tension and ρ is the density (for water $\gamma = 0.074\,\mathrm{N\,m^{-1}}$ and $\rho = 10^3\,\mathrm{kg\,m^{-3}}$ in MKS units).

Question 4 Perform similar calculations to those in Q3 for water using $\sigma = 10^{-2}$m. Contrast your results with those in Q3. *Inter alia* you will need to choose different values of the time-step and X.

Appendix: The Fast Fourier Transform

The "fast" in FFT depends on the number of modes, N, being a power of a prime, or combination of primes; for simplicity assume that $N = 2^M$. Write

$$\lambda_r \longleftrightarrow \mu_s \quad r, s = 0, \ldots (N-1) \tag{14}$$

to denote that (10) is satisfied. Introduce the half-length transforms

$$\left.\begin{array}{ccc} \lambda_r^E & \longleftrightarrow & \mu_{2s} \\ \lambda_r^O & \longleftrightarrow & \mu_{2s+1} \end{array}\right\} \quad r, s = 0, \ldots, (N/2-1) \ , \tag{15}$$

then it may be shown that

$$\left.\begin{array}{rcl}\lambda_r & = & \lambda_r^E + \Omega_N^{\sigma j}\lambda_r^O \\ \lambda_{r+N/2} & = & \lambda_r^E - \Omega_N^{\sigma j}\lambda_r^O\end{array}\right\} \quad r = 0, \ldots, (N/2-1)\ . \tag{16}$$

Hence if the half-length transforms are known, it costs $\frac{1}{2}N$ products to evaluate the λ_r.

To execute an FFT, start from N vectors of unit length (i.e. the original μ_s). At the s-th stage, $s = 1, 2, \ldots, M$, assemble 2^{M-s} vectors of length 2^s from vectors of length 2^{s-1} — this "costs" $2^{M-1} = \frac{1}{2}N$ products. The complete DFT has been formed after M stages, i.e. after $\mathcal{O}(\frac{1}{2}N\log_2 N)$ products. For $N = 1024 = 2^{10}$, say, the cost is $\approx 5 \times 10^3$ products, compared to $\approx 10^6$ products in naive matrix multiplication!

