2 Waves

2.7 Soliton Solutions of the KdV Equation (7 units)

This project assumes only basic knowledge of wave equations and does not rely directly on any Part II lecture course. The Part II(D) course Integrable Systems *may be useful but the project should be quite accessible to those who have not taken this course, using Drazin & Johnson [1] as a suitable reference.*

1 Theory

The Korteweg-de Vries (KdV) equation,

$$u_t + uu_x + \delta^2 u_{xxx} = 0\,, \tag{1}$$

arises in many branches of physics as a model for the evolution and interaction of nonlinear waves. It is well-known that the equation possesses single-soliton solutions of the form $u(x,t) = f(x - ct)$, where

$$f(x) = A\text{sech}^2\left(\frac{x - x_0}{\Delta}\right)\,, \quad \Delta^2 = 12\delta^2/A\,, \quad c = A/3\,, \tag{2}$$

and where A and x_0 are arbitrary constants representing the amplitude and initial location of the soliton respectively. It is supposed that $u(x,t)$ obeys the cyclic boundary conditions,

$$u(x+1,t) = u(x,t)\,, \tag{3}$$

so that only the region $0 \leqslant x \leqslant 1$ need be considered.

2 Questions

Question 1 Verify that the single soliton (2) is indeed a [non-periodic] solution of the KdV equation. For a periodic solution satisfying (3), prove that the mass, M, and the energy, E, of the motion, defined by

$$M \equiv \int_0^1 u(x,t)dx \quad \text{and} \quad E \equiv \int_0^1 \frac{1}{2}u^2(x,t)dx\,, \tag{4}$$

are independent of time.

Question 2 A leap-frog scheme first employed by Zabusky & Kruskal for solving the KdV equation is given on page 184 of Drazin & Johnson [1]. Let u_m^n be the solution at $x = hm$ and $t = kn$ with n, $m = 0$, $1, \ldots$, then the KdV equation can be discretised as

$$u_m^{n+1} = u_m^{n-1} - \frac{k}{3h}(u_{m+1}^n + u_m^n + u_{m-1}^n)(u_{m+1}^n - u_{m-1}^n) - \frac{k\delta^2}{h^3}(u_{m+2}^n - 2u_{m+1}^n + 2u_{m-1}^n - u_{m-2}^n)\,. \tag{5}$$

What is the order of this scheme? For $\delta = 1$, this scheme will be stable provided that

$$k \leqslant \frac{h^3}{4 + h^2|u_{\max}|}\,, \tag{6}$$

where $|u_{\max}|$ is the maximum modulus of u. By rescaling the KdV equation, derive the stability condition for $\delta \neq 1$.

Write a program to implement (5). Note that you will need an alternative method for making the first step; explain your choice carefully in your write-up. As as a check on the accuracy of your program, use it to calculate $u(x, 0.75)$ for the initial data

$$u(x,0) = A\,\mathrm{sech}^2\left(\frac{x-x_0}{\Delta}\right), \tag{7}$$

with $\delta = 0.04$, $A = 2.5$ and $x_0 = 0.25$.

Estimate the error in your results, and clearly explain the reasoning behind your choice of the values of k and h. Plot on the same axes graphs of both your numerical solution and the exact solution at $t = 0.75$ Comment on any differences. Does the propagation speed of the numerical solution agree exactly with that of the analytical solution?

Question 3 Describe and comment on the evolution of the initial data corresponding to the sum of two solitons, one with $A = 2.5$ and $x_0 = 0.25$, and a second with $A = 1$ and $x_0 = 0.75$, again with $\delta = 0.04$. Illustrate your answer with plots of the numerical solution at several instants. In addition, use the trapezium rule to calculate the mass and energy associated with your numerical solution at each time step, and comment on their variation with time (you need not output these quantities at every time step).

Question 4 Consider now

$$u(x,0) = \sin(2\pi x)\,. \tag{8}$$

In the case $\delta = 0$ give a very brief qualitative description of the evolution of this initial data. Use your program to investigate the case $\delta = 0.04$ numerically, plotting graphs of the solution at suitable instants. With particular reference to the relative magnitudes of the various terms in the KdV equation, explain why your numerical solution changes character at a certain time; estimate this time. Describe how the solution evolves after this time.

Try different values of δ, both larger and smaller than 0.04, and comment on the results that you obtain.

References

[1] Drazin, P.G. & Johnson, R.S. (1989) *Solitons: An Introduction*, CUP.

[2] Zabusky, N. & Kruskal, M (1965) Interaction of "solitons" in a collisionless plasma and the recurrence of initial states. *Physical Review Letters*, volume 15, number 6, pages 240-3.

