3 Fluid and Solid Dynamics

3.1 Boundary-Layer Flow (6 units)

This project is based on a section of the Part II course Fluid Dynamics but the relevant material can be found in Chapter 8 of [1] or Chapter V of [2]; a brief outline is given below.

1 Background Theory

Consider flow of an incompressible viscous fluid of constant density ρ and kinematic viscosity ν, whose velocity $\mathbf{u}(\mathbf{x}, t)$ and hydrodynamic pressure $p(\mathbf{x}, t)$ satisfy the incompressibility condition

$$\nabla.\mathbf{u} = 0\,, \tag{1}$$

and the Navier-Stokes momentum equation

$$\rho\left(\mathbf{u}_t + (\mathbf{u}.\nabla)\,\mathbf{u}\right) = -\nabla p + \rho\nu\nabla^2\mathbf{u}\,. \tag{2}$$

It follows that the vorticity $\boldsymbol{\omega} \equiv \nabla \times \mathbf{u}$ satisfies

$$\boldsymbol{\omega}_t + (\mathbf{u}.\nabla)\,\boldsymbol{\omega} - (\boldsymbol{\omega}.\nabla)\,\mathbf{u} = \nu\nabla^2\boldsymbol{\omega}\ . \tag{3}$$

If the flow is two-dimensional, it can be expressed in terms of a streamfunction $\psi(x, y, t)$, with

$$\mathbf{u} \equiv (u, v, 0) = \left(\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}, 0\right)\ , \quad \boldsymbol{\omega} = \left(0, 0, -\nabla^2\psi\right)\ , \tag{4}$$

in terms of which the vorticity equation (3) becomes

$$\frac{\partial}{\partial t}\left(\nabla^2\psi\right) - \frac{\partial\left(\psi, \nabla^2\psi\right)}{\partial\,(x, y)} = \nu\nabla^2\left(\nabla^2\psi\right)\ . \tag{5}$$

Suppose there is a stationary rigid boundary on $y = 0$, $x > 0$ on which must be satisfied the conditions of no slip, $u = 0$, and no penetration, $v = 0$. If the viscosity is ‘small’, then away from the boundary the flow *may*, to a good approximation, be irrotational: in this case $\mathbf{u} \approx \nabla\phi$ for a velocity potential ϕ satisfying $\nabla^2\phi = 0$ (incompressibility) and $\partial\phi/\partial y = 0$ on $y = 0$, $x > 0$ (no penetration). These conditions, together with corresponding ones on other boundaries and/or at infinity, are sufficient to determine $\nabla\phi$ uniquely. As a result, it is not possible to specify the tangential (‘slip’) velocity component on the boundary, i.e.

$$\frac{\partial\phi}{\partial x}(x, 0, t) \equiv U_e(x, t) \quad (x > 0)\,.$$

Instead, $U_e(x, t)$ is determined as part of the irrotational potential-flow solution, and is necessarily non-zero. There must therefore be a ‘boundary layer’ near $y = 0$ where the potential-flow approximation is not valid and satisfaction of the no-slip condition implies viscous diffusion of vorticity away from the boundary.

If the boundary layer is ‘thin’, in the sense that within it $\partial/\partial x \ll \partial/\partial y$ (i.e. variations with respect to y are much more rapid than variations with respect to x), then the approximations

$$\nabla^2\psi \approx \psi_{yy}\ , \quad \nabla^2\left(\nabla^2\psi\right) \approx \psi_{yyyy}\,,$$

may be made in equation (5), which can then be integrated once with respect to y to give

$$\psi_{yt} + \psi_y \psi_{xy} - \psi_x \psi_{yy} = G(x,t) + \nu \psi_{yyy} \ , \tag{6}$$

This is in fact the x-component of the Navier-Stokes momentum equation (2) with the $\rho\nu u_{xx}$ term neglected (since it is small compared to $\rho\nu u_{yy}$). The forcing term, or more precisely $-\rho G$, may be identified with the pressure gradient $\partial p/\partial x$ (which in this approximation is independent of y, i.e. uniform across the 'thin' boundary layer). Equation (6) is to be solved in conjunction with the conditions of no slip and no penetration at the boundary, i.e.

$$\psi_y = 0\,, \quad -\psi_x = 0 \ \Rightarrow \ \psi = \psi_0(t)\,, \quad \text{on } y = 0\,, \tag{7}$$

where $\psi_0(t) = 0$ without loss of generality. In addition the solution must 'match' to the outer potential flow, i.e.

$$u \equiv \psi_y \to U_e(x,t) \quad \text{as } y \to \infty\,. \tag{8}$$

It follows from (6) and (8) that

$$G(x,t) = \frac{\partial U_e}{\partial t} + U_e \frac{\partial U_e}{\partial x} \ . \tag{9}$$

In the special case when the flow is steady, $\partial/\partial t = 0$, and when

$$U_e(x) = A x^m \quad \text{for } x > 0\,, \tag{10}$$

with A and m constants, the problem (6)–(9) admits a 'similarity' solution with

$$\psi(x,y) = |U_e(x)|\delta(x) f(\eta) \ , \quad \eta = \frac{y}{\delta(x)} \quad \text{and} \quad \delta(x) = \left(\frac{\nu x}{|U_e(x)|}\right)^{\frac{1}{2}} \ . \tag{11}$$

Here, $\delta(x)$ is a measure of the local boundary-layer thickness, and the function f satisfies the *Falkner–Skan* equation

$$m\,(f')^2 - \tfrac{1}{2}(m+1) f f'' = m + f''' \,, \tag{12}$$

with the boundary conditions

$$f' = f = 0 \quad \text{on } \eta = 0, \qquad f' \to \operatorname{sgn} A \quad \text{as } \eta \to \infty. \tag{13}$$

In fact, if there is no source of vorticity other than the boundary $y = 0$, then f' should converge to $\operatorname{sgn} A$ *exponentially fast* as $\eta \to \infty$, i.e.

$$\eta^N \left(f' - \operatorname{sgn} A\right) \to 0 \quad \text{as } \eta \to \infty \quad \text{for any } N. \tag{14}$$

The tangential velocity component is obtained from (4) and (11) as

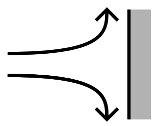
$$u = \frac{\partial \psi}{\partial y} = |U_e(x)| f'(\eta)\,. \tag{15}$$

The tangential stress (force per unit area in the x-direction) exerted by the fluid on the boundary $y = 0$ is

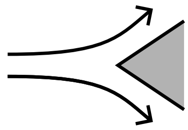
$$\tau_0 \equiv \rho\nu \left.\frac{\partial u}{\partial y}\right|_{y=0} = \rho \left(\frac{\nu |U_e(x)|^3}{x}\right)^{\frac{1}{2}} f''(0). \tag{16}$$

Different values of m arise for different external flows. Particular cases are (taking $A > 0$ unless stated otherwise):

$m = 1$: Flow towards a stagnation point on a plane wall.

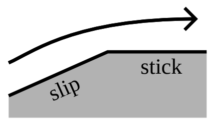


$0 < m < 1$: Flow past a wedge of semi-angle $\theta = \frac{\pi m}{m+1}$.

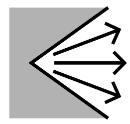


$m = 0$: Flow past a flat plate (the Blasius boundary layer).

$-\frac{1}{2} < m < 0$: Flow around the outside of a corner, turning through an angle $\theta = -\frac{\pi m}{m+1}$ with slip (but no penetration) upstream of the corner, and no slip (and no penetration) downstream. This is an artificial problem, but it *might* be that the solution could arise as the downstream limit of a realistic flow.



$m = -1$: Flow due to a line source (for $A > 0$) or line sink ($A < 0$) at the intersection of two plane rigid boundaries (at arbitrary angle).



The aim of this project is to solve the two-point-boundary-value problem (12)–(14) by 'shooting', finding by trial-and-error the values of $f''(0)$ which give a solution with the required behaviour as $\eta \to \infty$. Except in the last question, attention is to be restricted to the case $A > 0$.

2 Analysis

Question 1 Examine analytically the possibility that a solution of the Falkner-Skan equation (12) has one of the following terminal behaviours:

(i) algebraic convergence,

$$f' \sim 1 + B\eta^{-k} \quad \text{as } \eta \to \infty\,;$$

(ii) exponential convergence,

$$f \sim \eta - \eta_0 + \mathrm{e}^{-\sigma(\xi)} \quad \text{as } \eta \to \infty\,,$$

where $\xi = \eta - \eta_0$ and

$$\sigma'(\xi) = k\xi + k' + k''\xi^{-1} + O\left(\xi^{-2}\right);$$

(iii) algebraic divergence,

$$f \sim B\eta^{1+k} \quad \text{as } \eta \to \infty\,;$$

(iv) exponential divergence,

$$f \sim B\mathrm{e}^{k\eta} \quad \text{as } \eta \to \infty\,;$$

(v) a finite-distance singularity, at $\eta = \eta_0$ say, with

$$f \sim B\left(\eta_0 - \eta\right)^{-1} \quad \text{as } \eta \to \eta_0\,.$$

Here B, η_0, k, k', k'' are constants with $B \neq 0$ and $k > 0$. For each case deduce (a) which (if any) of these constants can be determined without use of conditions at $\eta = 0$, and (b) what restrictions (if any) must be placed on the value of m for this type of behaviour to be possible.

3 Computation

Programming Task: Write a program to integrate the Falkner–Skan equation (12) subject to the one-point boundary conditions

$$f(0) = f'(0) = 0 \ , \quad f''(0) = S \, , \tag{17}$$

where S is a known constant. You may use a black-box numerical integrator for this problem such as the MATLAB routine **ode45**, but you should not use any *two-point*-boundary-value solver such as the MATLAB routine **bvp4c** (except possibly as a check).

Question 2 Integrate the Falkner-Skan equation (12) with boundary conditions (17) for $m = 0$ and $S = 1$. You should find that f' converges to a constant as $\eta \to \infty$; comment on the nature of the convergence, and determine the constant to at least four significant figures, presenting evidence that this accuracy has been achieved.

Explain how it is possible to deduce, without further computation, a solution of the boundary-value problem (12)–(13) for $m = 0$.

Hint: consider $af(b\eta)$ for suitable constants a and b.

Question 3 Now apply the shooting method for $0 \leqslant m \leqslant 1$. You should find that for each m in this range, there is a unique value of S, call it S_m, for which $f' \to 1$ as $\eta \to \infty$. For $m = \frac{2}{5}$ and $m = 1$, plot graphs of f' against η for various values of S, both less than and greater than S_m, and comment on their terminal behaviour.

Write a program to determine S_m. Note:

- you may wish to use a black-box root-finder such as the MATLAB routine `fzero`;
- it may also be helpful, for the next question, to have the option of finding the inverse, i.e. determining m for given S.

Tabulate and plot S_m against m for $0 \leqslant m \leqslant 1$ correct to at least four significant figures, explaining why you are satisfied with the accuracy.

Comment on the *physical* interpretation of your solutions, e.g. the effect of varying m.

Question 4 Investigate solutions of the Falkner-Skan equation (12) subject to (17) for various m in the range $-1 < m < 0$ and various S (both positive and negative), and display some representative results, illustrating the different kinds of terminal behaviour which occur. In the range $m_c < m < 0$ (where m_c is to be determined) you should find two branches of exponentially converging solutions, one of which is the continuation of that already found for $m \geqslant 0$. Tabulate and plot S_m against m for both branches, and plot f' for both solutions against η for at least one value of m. Discuss the *physical* interpretation of each solution, and the form of the second solution as $m \uparrow 0$.

In the interval $-1 < m < m_c$, there are other branches of exponentially converging solutions: plot at least two of these branches in the m-S plane, and present graphs of f' against η for a few of the solutions. Comment on their *physical* significance.

Question 5 Investigate (numerically) $m = -1$, for both signs of A. What are your conclusions?

References

[1] Acheson, D. J., *Elementary Fluid Dynamics.* O.U.P.

[2] Rosenhead, L. (ed.), *Laminar Boundary Layers.* Dover. (In particular Chapter V, sections 1, 12-17, 21.)

[3] Sobey, I. J., *Introduction to Interactive Boundary Layer Theory.* O.U.P.

[4] Stewartson, K. (1954) Further solutions of the Falkner-Skan equation. *Proceedings of the Cambridge Philosophical Society*, volume 50, issue 03, pages 454-465. See http://dx.doi.org/10.1017/S030500410002956X