4 Dynamics

4.3 The Rotating Top (6 units)

This project assumes knowledge of the relevant material from the Classical Dynamics course, which may also be found in the reference listed.

This project concerns the familiar problem of the axisymmetric top, rotating about a fixed point on its axis of symmetry. By choosing units such that $mgh = A$, the Lagrangian (scaled by A) is

$$L = \tfrac{1}{2}[\dot\theta^2 + \dot\phi^2 \sin^2\theta] + \tfrac{1}{2}C[\dot\psi + \dot\phi\cos\theta]^2 - \cos\theta.$$

Here θ, ϕ, ψ are the usual 3 Eulerian angles. C is the ratio of the principal moments of inertia (in usual, dimensional, notation, equal to C/A). The first Lagrangian equation of motion is then

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) - \frac{\partial L}{\partial\theta} = \ddot\theta - \dot\phi^2\sin\theta\cos\theta + C\dot\phi\sin\theta\,[\dot\psi + \dot\phi\cos\theta] - \sin\theta.$$

The other two can immediately be integrated once to give

$$\frac{\partial L}{\partial\dot\psi} = \text{constant} \qquad \Rightarrow \qquad C[\dot\psi + \dot\phi\cos\theta] = \alpha, \tag{1}$$

$$\frac{\partial L}{\partial\dot\phi} = \text{constant} \qquad \Rightarrow \qquad \alpha\cos\theta + \dot\phi\sin^2\theta = \beta. \tag{2}$$

The energy integral is

$$E = \tfrac{1}{2}[\dot\theta^2 + \dot\phi^2\sin^2\theta] + \tfrac{1}{2}C[\dot\psi + \dot\phi\cos\theta]^2 + \cos\theta = \text{constant}. \tag{3}$$

Further details of the theory can be found in §5.7 of [1].

The equations can be rewritten in a form suitable for numerical solution as

$$\dot\phi = \frac{\beta - \alpha\cos\theta}{\sin^2\theta}, \tag{4}$$

$$\ddot\theta = [\dot\phi(\dot\phi\cos\theta - \alpha) + 1]\sin\theta. \tag{5}$$

Programming Task: Write a program to investigate the motion numerically and to display the motion graphically. You may use any numerical method which you consider suitable, or library routines. For each of the questions below, you should consider the most appropriate and informative form of graphical output; possibilities include (but are not limited to) either 3D plots or 2D plots such as θ vs. ϕ, θ vs. t or ϕ vs. t.

Comment on the numerical method chosen. Check the accuracy of your program by printing out E during each run, and comment on your results.

What problem can arise when $\sin\theta$ becomes small? Your program will need to include a method for dealing with this difficulty, and you should check that it produces accurate results.

In later questions you may find that θ sometimes leaves its usual domain ($0 \leqslant \theta \leqslant \pi$). Should this occur, explain the physical meaning of your results and how your computed values for θ are related to the proper values.

Question 1 Explain the general theory underlying the motion of a rotating top, starting from equations (1)–(3). Demonstrate that your program can simulate the three main types of motion below and explain what initial conditions are required.

1. Normal precession ($\dot{\phi}$ does not change sign)
2. Retrograde motion ($\dot{\phi}$ changes sign)
3. Motion with cusps (the border between 1 and 2)

You should obtain one hard copy of each type of motion.

Question 2 Choose values for α, θ such that there are 2 values of $\dot{\phi}$ that give a solution with constant θ, and show that your program can replicate the motion. Explain how these results fit the general theory.

Question 3 Investigate the stability of a sleeping top (i.e., one that spins with $\theta = 0$) by giving the motion a small disturbance; specifically, use initial conditions $\theta = 0$ and $\dot{\theta}$ non-zero but small. Explain clearly the possible types of subsequent motion and give an appropriate criterion for stability. Obtain a rough estimate for the critical value of α, which should be independent of C, and show examples of the motion just above and below the critical value. Is your estimate consistent with the theoretical predictions?

Question 4 Take α, θ very small (both 0.01 say) with initial conditions $\dot{\theta} = \dot{\phi} = 0$. What happens? Give a physical interpretation. Now change α to zero, and explain your results.

References

[1] H. Goldstein, *Classical Mechanics*, 2nd edition. Addison-Wesley, 1980.

