6 Electromagnetism

6.1 Diffraction pattern due to a current strip (7 units)

Knowledge of material covered in the Part IB course Electromagnetism is useful as background.

The object of this project is to investigate the magnetic field generated by a strip current alternating at radio frequency.

1 Theory

Consider current flowing in an infinite 2-dimensional strip. Let us choose our coordinate axes such that current is nonzero in the 2-dimensional surface defined by $-d < x < d$, $y = 0$, and all z. The current flows in the z-direction and oscillates in time so that its strength j is given by

$$j = e^{i\omega t}.$$

The wavelength associated with this oscillation is given by $\lambda = 2\pi c/\omega$. Suppose that there is an integral number n of wavelengths in the current strip, i.e., $2d = n\lambda$; then all lengths may be normalised so that $\lambda = 1$. It may be shown that the component of the magnetic field in the x-direction is given by $e^{i\omega t} H_x(x, y)$, where

$$H_x(x,y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\sin(n\pi l)}{l} \exp\left\{2\pi i \left(lx + y\sqrt{1-l^2}\right)\right\} dl. \tag{1}$$

For large y, the magnetic field asymptotically tends to

$$|H_x| \sim \frac{1}{2\pi} \sqrt{1-p^2}\, \frac{|\sin n\pi p|}{\sqrt{xp}} \tag{2}$$

where

$$p = \frac{x}{\sqrt{x^2+y^2}}.$$

2 Method

The right-hand side of (1) is a Fourier type integral and may be approximated by a discrete Fourier transform. Truncate the integration at $|l| = l_{\max}$, so that (1) becomes of the form

$$H_x \sim \frac{1}{2\pi} \int_{-l_{\max}}^{l_{\max}} a(l) e^{2\pi i l x} dl. \tag{3}$$

'Periodise' $a(l)$ so that

$$\begin{aligned} a_p(l) &= a_p(l - 2r l_{\max}) && \text{for any integer } r, \\ a_p(l) &= a(l) && \text{when } |l| < l_{\max}. \end{aligned}$$

Then the N-point discrete Fourier transform which approximates to (3) should be computed from the complex array of sample points

$$a_r = a_p(r\Delta l), \qquad r = 0, \ldots, N-1$$

where $\Delta l = 2l_{\max}/N$; specifically

$$H_x(s\Delta x) \equiv H_s \approx \frac{\Delta l}{2\pi} \sum_{r=0}^{N-1} a_r e^{2\pi i r s/N} \tag{4}$$

where $\Delta x = 1/(2l_{\max})$. Define $X_{\max} = \frac{1}{2}N\Delta x = N/(4l_{\max})$.

The discrete Fourier transform is best evaluated by the Fast Fourier Transform (FFT) method. You may use the Matlab routine `fft` or an equivalent routine in any other language, or you may write your own routine (but do not simply compute (4) directly). The FFT method is described in the Appendix, but it is not necessary to understand any of the details; it is sufficient simply to invoke the routine.

3 Questions

All your program output should be in graphical form. Plot graphs of:

(i) the real and imaginary parts of the whole array a_r;

(ii) the real and imaginary parts of the whole array H_s;

(iii) the modulus of the array H_s (either with (ii) or separately);

against l or x as appropriate. This is useful as it enables you to check by eye that your results are showing the correct symmetry properties.

Question 1 Run your program with the following values, making hard copies of your results.

n	y	$X_{\max}$	N
2	0.04	3	256
2	0.2	5	256
2	10	25	128
2	10	50	256
4	5	50	256
4	10	50	256
4	25	50	256

Explain how the results are affected by the parameters $X_{\max}$ and N at different values of y. (You may naturally want to try other values than in the table, in order to develop your hypotheses.)

Question 2 Modify your program so that it can produce, on one graph, plots of $|H_x|$ against x for $y = 0.1$, 1, 5, 10, and 25 with $N = 256$, using suitable values of $X_{\max}$ for each y. Produce a hard copy of your results for $n = 2$, 3, 4.

Question 3 Comment on how you ensured the accuracy of the curves plotted in Question 2.

Question 4 Plot equation (2) against x for some of the same values of y as in the previous question. Decide on the best way to present these plots so that you can justify any statements you make comparing the results using (2) vs. (3).

Question 5 Comment on the physical significance of your results. In particular, how do your results demonstrate the phenomena of diffraction?

Appendix: The Fast Fourier Transform

The Fast Fourier Transform (FFT) technique is a quick method of evaluating sums of the form

$$\lambda_r = \sum_{s=0}^{N-1} \mu_s \, \omega_N^{\sigma r s}, \qquad r = 0, \ldots, N-1, \qquad \sigma = \pm 1, \tag{5}$$

where N is an integer, μ_s is a known sequence and $\omega_N = e^{2\pi i/N}$. The "fast" in FFT depends on N being a power of a small prime, or combination of small primes; for simplicity we will assume that $N = 2^M$. Write

$$\lambda_r \longleftrightarrow \mu_s, \qquad r, s = 0, \ldots, N-1$$

to denote that (5) is satisfied. Introduce the half-length transforms

$$\left.\begin{aligned} \lambda_r^E &\longleftrightarrow \mu_{2s} \\ \lambda_r^O &\longleftrightarrow \mu_{2s+1} \end{aligned}\right\} \qquad r, s = 0, \ldots, \tfrac{1}{2}N - 1;$$

then it may be shown that

$$\left.\begin{aligned} \lambda_r &= \lambda_r^E + \omega_N^{\sigma r} \lambda_r^O \\ \lambda_{r+N/2} &= \lambda_r^E - \omega_N^{\sigma r} \lambda_r^O \end{aligned}\right\} \qquad r = 0, \ldots, \tfrac{1}{2}N - 1.$$

Hence if the half-length transforms are known, it costs $\frac{1}{2}N$ products to evaluate the λ_r.

To execute an FFT, start from N vectors of unit length (i.e., the original μ_s). At the sth stage, $s = 1, 2, \ldots, M$, assemble 2^{M-s} vectors of length 2^s from vectors of length 2^{s-1} – this "costs" $2^{M-s} \times \frac{1}{2}(2^s) = 2^{M-1} = \frac{1}{2}N$ products for each stage. The complete discrete Fourier transform has been formed after M stages, i.e., after $O(\frac{1}{2}N \log_2 N)$ products. For $N = 1024 = 2^{10}$, say, the cost is $\approx 5 \times 10^3$ products, compared to $\approx 10^6$ products in naive matrix multiplication!

A description and short history of the FFT are given in the book *Numerical Recipes* by Press *et al.*, chapter 12.

